SOLUTION OF THE MIXED BOUNDARY VALUE PROBLEM OF HEAT CONDUCTION IN A COMPOSITE RECTANGLE

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The author offers a particular solution of the two-dimensional problem of steady heat conduction for a rectangular region composed of two different materials with mixed boundary conditions.

The following problem arises in connection with the problem of steady heat conduction in insulation systems with through inclusions:

$$\frac{\partial^2 t_1}{\partial x^2}+\frac{\partial^2 t_1}{\partial y^2}=0 \quad \begin{pmatrix} -l_1\leqslant x\leqslant 0 \\ 0\leqslant y\leqslant R \end{pmatrix}, \tag{1}$$

$$\frac{\partial^2 t_2}{\partial x^2}+\frac{\partial^2 t_2}{\partial y^2}=0 \quad \begin{pmatrix} 0\leqslant x\leqslant l_2 \\ 0\leqslant y\leqslant R \end{pmatrix} \tag{2}$$

with boundary conditions at $y=0$

$$t_1=t_0=\mathrm{const} \tag{3}$$

and

$$t_2=t_0=\mathrm{const}; \tag{4}$$

at $y=R$ there is heat transfer to a medium at zero temperature

$$\frac{\partial t_1}{\partial y}+h_1 t_1=0, \tag{5}$$

$$\frac{\partial t_2}{\partial y}+h_2 t_2=0; \tag{6}$$

at $x=-l_1$

$$\frac{\partial t_1}{\partial x}=0, \tag{7}$$

at $x=l_2$

$$\frac{\partial t_2}{\partial x}=0 \tag{8}$$

by virtue of the symmetry of the selected element; at $x=0$

$$t_1=t_2 \tag{9}$$

and

$$\lambda_1\frac{\partial t_1}{\partial x}=\lambda_2\frac{\partial t_2}{\partial x}, \tag{10}$$

i.e., the thermal contact is ideal.

To solve the problem we apply a Fourier sine transformation with respect to y with finite limits [1].

For the first rectangle (1)

$$T_1(x,\ \nu_n)=\int_0^R t_1(x,\ y)\sin\nu_n y\,dy. \tag{11}$$

For the transform differential equation (1) and condition (5) take the following form:

$$\frac{d^2T_1}{dx}-\nu_n^2 T_1=-\nu_n t_0, \tag{12}$$

$$\left(\frac{dT_1}{dx}\right)_{x=-l_1}=0. \tag{13}$$

In this case the ν_n are roots of the equation

$$h_1=-\nu_n\operatorname{ctg}\nu_n R. \tag{14}$$

The solution of Eq. (12) with condition (13) will be

$$T_1=\frac{t_0}{\nu_n}+B_1\frac{\operatorname{ch}\nu_n(l_1+x)}{\operatorname{sh}\nu_n l_1}, \tag{15}$$

where B_1 is a constant of integration.

By analogy

$$T_2=\frac{t_0}{\mu_n}-B_2\frac{\operatorname{ch}\mu_n(l_2-x)}{\operatorname{sh}\mu_n l_2}, \tag{16}$$

where B_2 is a constant of integration, and μ_n are roots of the equation

$$h_2=-\mu_n\operatorname{ctg}\mu_n R. \tag{17}$$

The inversion formula for our problem has the form [1]

$$t=\sum_{n=1}^{\infty}\frac{T}{I_n}\sin\nu_n y, \tag{18}$$

and

$$2\,\frac{\nu_n^2+h^2}{R\nu_n^2+Rh^2+h}=\frac{1}{I_n}. \tag{19}$$

Applying the inversion formula, we obtain the solution for rectangles 1 and 2:

$$t_1=\sum_{n=1}^{\infty}\frac{t_0}{\nu_n}\,\frac{\sin\nu_n y}{I_{n1}}+\sum_{n=1}^{\infty}B_{n1}\,\frac{\operatorname{ch}\nu_n(l_1+x)}{\operatorname{sh}\nu_n l_1}\,\frac{\sin\nu_n y}{I_{n1}}, \tag{20}$$

$$t_2=\sum_{n=1}^{\infty}\frac{t_0}{\mu_n}\,\frac{\sin\mu_n y}{I_{n2}}-\sum_{n=1}^{\infty}B_{n2}\,\frac{\operatorname{ch}\mu_n(l_2-x)}{\operatorname{sh}\mu_n l_2}\,\frac{\sin\mu_n y}{I_{n2}}. \tag{21}$$

It should be noted that the first series in (20) and (21) can be represented in closed form [2]; then

$$\sum_{n=1}^{\infty}\frac{t_0}{\nu_n}\,\frac{\sin\nu_n y}{I_{n1}}=f_1(y)=t_0\left(1-\frac{h_1R}{1+h_1R}\,\frac{y}{R}\right), \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{t_0}{\mu_n} \frac{\sin \mu_n y}{I_{n2}} = f_2(y) =$$

$$= t_0 \left(1 - \frac{h_2 R}{1 + h_2 R} \frac{y}{R}\right). \qquad (23)$$

The solution for the two rectangles, using conditions (3)–(8) at the three boundaries is as follows:

$$t_1 = f_1(y) + \sum_{n=1}^{\infty} B_{n1} \frac{\operatorname{ch} \nu_n (l_1 + x)}{\operatorname{sh} \nu_n l_1} \frac{\sin \nu_n y}{I_{n1}}, \qquad (24)$$

$$t_2 = f_2(y) - \sum_{n=1}^{\infty} B_{n2} \frac{\operatorname{ch} \mu_n (l_2 - x)}{\operatorname{sh} \mu_n l_2} \frac{\sin \mu_n y}{I_{n2}} \qquad (25)$$

The condition at the boundary x = 0 makes it possible to determine the constants B_{n1} and B_{n2} from the system

$$\begin{cases} \lambda_1 \sum_{n=1}^{\infty} B_{n1} \frac{\nu_n}{I_{n1}} \sin \nu_n y = \\ = \lambda_2 \sum_{n=1}^{\infty} B_{n2} \frac{\mu_n}{I_{n2}} \sin \mu_n y, \\ f_1(y) + \sum_{n=1}^{\infty} B_{n1} \frac{\operatorname{cth} \nu_n l_1}{I_{n1}} \sin \nu_n y = \\ = f_2(y) - \sum_{n=1}^{\infty} B_{n2} \frac{\operatorname{cth} \mu_n l_2}{I_{n2}} \sin \mu_n y. \end{cases} \qquad (26)$$

However, this leads to the necessity of determining the constants B_n from an infinite system of infinite equations.

To avoid this difficulty, we assume that the temperature distribution at the boundary of the two materials (at x = 0) is linear.* In this case, by representing the temperature distribution along the contact line in general form, we can formally regard it as specifying the boundary conditions for both rectangles at the boundary x = 0, i.e., we can assume that

$$t_1 = f(y) = t_2 = t_0 (1 - by). \qquad (27)$$

then

$$t_1 = f_1(y) + \sum_{n=1}^{\infty} \int_0^R [f_1(y) - f(y)] \sin \nu_n y dy \times$$

$$\times \frac{\operatorname{ch} \nu_n (l_1 + x)}{\operatorname{ch} \nu_n l_1} \frac{\sin \nu_n y}{I_{n1}}, \qquad (28)$$

$$t_2 = f_2(y) - \sum_{n=1}^{\infty} \int_0^R [f_2(y) - f(y)] \sin \mu_n y dy \times$$

$$\times \frac{\operatorname{ch} \mu_n (l_2 - x)}{\operatorname{ch} \mu_n l_2} \frac{\sin \mu_n y}{I_{n2}}. \qquad (29)$$

In solution (28), (29) only the gradient remains undetermined.

The requirements of the boundary condition on the line of contact between the two materials with respect to the heat-flux equality (10) enables us to determine the gradient

$$b = \left[\alpha \left(\sum_{n=1}^{\infty} \operatorname{th} \nu_n l_1 \frac{\sin \nu_n R}{\nu_n} \frac{\sin \nu_n y}{I_{n1}} + \right.\right.$$

$$\left.\left. + \sum_{n=1}^{\infty} \operatorname{th} \mu_n l_2 \frac{\sin \mu_n R}{\mu_n} \frac{\sin \mu_n y}{I_{n2}}\right)\right] \times$$

$$\times \left[\lambda_1 (1 + h_1 R) \sum_{n=1}^{\infty} \operatorname{th} \nu_n l_1 \frac{\sin \nu_n R}{\nu_n} \frac{\sin \nu_n y}{I_{n1}} + \right.$$

$$\left. + \lambda_2 (1 + h_2 R) \sum_{n=1}^{\infty} \operatorname{th} \mu_n l_2 \frac{\sin \mu_n R}{\mu_n} \frac{\sin \mu_n y}{I_{n2}}\right]^{-1}. \qquad (30)$$

Satisfying condition (10) introduced a refinement into assumption (27) and made it possible to obtain a method of determining the temperature field of the composite rectangle.

Thus, we finally obtain

$$\frac{t_1}{t_0} = (1 - a_1 y) + (a_1 - b)(1 + h_1 R) \sum_{n=1}^{\infty} \frac{\sin \nu_n R}{\nu_n^2 I_{n2}} \times$$

$$\times \frac{\operatorname{ch} \nu_n (l_1 + x)}{\operatorname{ch} \nu_n l_1} \sin \nu_n y, \qquad (31)$$

$$\frac{t_2}{t_0} = (1 - a_2 y) + (a_2 - b)(1 + h_2 R) \sum_{n=1}^{\infty} \frac{\sin \mu_n R}{\mu_n^2 I_{n2}} \times$$

$$\times \frac{\operatorname{ch} \mu_n (l_2 - x)}{\operatorname{ch} \mu_n l_2} \sin \mu_n y, \qquad (32)$$

where b is found from (30). The roots ν_n and μ_n are presented in [3] (Table 71).

The solution for each region is an algebraic sum. One term is the solution for the corresponding infinite plate, while the second expresses the flow of heat along the heated surface due to contact with the other material.

For practical calculations it is more convenient to use the mean-integral gradient of the temperature distribution at the boundary of the two materials, which is obtained from the equation

$$\int_0^R \lambda_1 \left(\frac{\partial t_1}{\partial x}\right)_{x=0} dy = \int_0^R \lambda_2 \left(\frac{\partial t_2}{\partial x}\right)_{x=0} dy, \qquad (33)$$

$$b = \left\{\alpha \left[\sum_{n=1}^{\infty} \frac{\operatorname{th} \nu_n l_1}{I_{n1}} \frac{\sin \nu_n R}{\nu_n^2} (1 - \cos \nu_n R) + \right.\right.$$

$$\left.\left. + \sum_{n=1}^{\infty} \frac{\operatorname{th} \mu_n l_2}{I_{n2}} \frac{\sin \mu_n R}{\mu_n^2} (1 - \cos \mu_n R)\right]\right\} \times$$

$$\times \left[\lambda_1 (1 + h_1 R) \sum_{n=1}^{\infty} \frac{\operatorname{th} \nu_n l_1 \sin \nu_n R}{I_{n1} \nu_n^2} (1 - \cos \nu_n R) + \right.$$

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* This assumption was based on the results of numerous temperature field calculations by the finite difference method on a computer and a MSM-1 and on the results of experiments on conductive paper. It may be assumed with an accuracy acceptable for practicle calculations that at x = 0 the function t = f(y) is linear.

$$+\lambda_2(1+h_2R)\times$$

$$\times\sum_{n=1}^{\infty}\frac{\text{th}\,\mu_n l_2\sin\mu_n R}{I_{n2}\mu_n^2}(1-\cos\mu_n R)\Big]^{-1}. \quad (34)$$

The figure shows the temperature distribution in the composite rectangular region as calculated from (31), (32), (34). For comparison the figure includes the results of a determination of the field on conductive paper for the following starting data: $\lambda_1 = 5.8$W/deg · m, $\alpha = 11.6$W/deg · m^2, $l_2 = 0.2$m; $\lambda_2 = 0.232$ W/deg · m, $l_1 = 0.02$m, R = 0.4m.

It should be noted that the solution obtained can be extended to the very common practical problem with somewhat more complicated boundary conditions.

The problem can be formulated completely by means of Eqs. (1) and (2) with conditions (5)–(10) and the following conditions, different from the previous problem, at the boundary y = 0:

$$\lambda_1\frac{\partial t_1}{\partial y}+\alpha_m(t_m-t_1)=0 \text{ when } -l_1\leqslant x\leqslant 0, \quad (I)$$

$$\lambda_2\frac{\partial t_2}{\partial y}+\alpha_m(t_m-t_2)=0 \text{ when } 0\leqslant x\leqslant l_2. \quad (II)$$

We assume that the solutions of Eqs. (1) and (2) will be the solutions for the corresponding infinite plates with conditions at the boundaries y = 0 (I, II) and y = R (5), (6) [4]:

$$t_1=t_m-\frac{\frac{y}{\lambda_1}+\frac{1}{\alpha_m}}{\frac{1}{\alpha_m}+\frac{R}{\lambda_1}+\frac{1}{\alpha}}t_m, \quad (35)$$

Boundary conditions (7), (8), (10) are then satisfied. In accordance with requirement (9), we equate (35) to (36). From this equality we obtain the value of y at

$$t_2=t_m-\frac{\frac{y}{\lambda_2}+\frac{1}{\alpha_m}}{\frac{1}{\alpha_m}+\frac{R}{\lambda_2}+\frac{1}{\alpha}}t_m. \quad (36)$$

which (35) and (36) satisfy the basic equations and all the boundary conditions. At

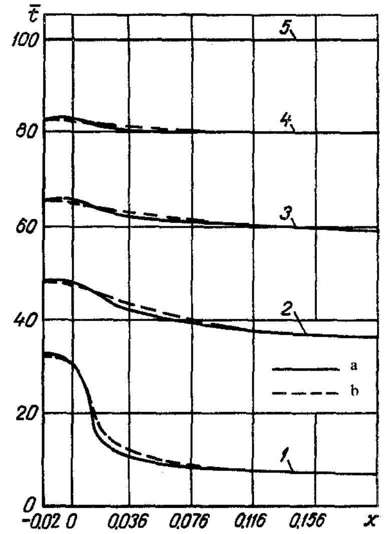
$$y=y_0=R\frac{\alpha}{\alpha+\alpha_m}$$

the temperature in the system does not depend on x and $t_1 = t_2 = t_0$ = const. The isotherm t_0 is parallel to the surfaces of the plate. The value of t_0 can always be found from (35) or (36).

The definite position and known value of this particular isotherm make it possible to divide the composite rectangle into two parts, for which the solution has already been found in (31), (32).

NOTATION

t_1 and t_2 denote the temperatures of the corresponding regions of the composite rectangle; x and y are



Temperature distribution in composite rectangle along x-axis ($\bar{t}$ in %, x in m, y in m): 1) y = 0.4; 2) 0.3; 3) 0.2; 4) 0.1); 5) 0; a) calculated by the proposed method; b) the same from the potential field of a model.

coordinates; λ_1 and λ_2 are the thermal conductivities of the corresponding regions; α is the heat-transfer coefficient; $h_1 = \alpha/\lambda_1$, $h_2 = \alpha/\lambda_2$; ν_n and μ_n are integral transform parameters; T_1 and T_2 are respective transforms of the functions $t_1(x, y)$ and $t_2(x, y)$; $a_1 = h_1/(1 + h_1R)$, $a_2 = h_2/(1 + h_2R)$; t_m is the temperature of surrounding medium on the side of the rectangle y = 0; α_m is the coefficient of thermal absorption.

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